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Completions of Leavitt path algebras

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Abstract We introduce a class of topologies on the Leavitt path algebra $L(\Gamma)$ of a finite directed graph and decompose a graded completion $\widehat{L}(\Gamma)$ as a direct sum of minimal ideals.

Keywords Associative algebra · Leavitt path algebra · Topological algebra

1 Definitions and terminology

The purpose of this paper is to define a class of linear topologies on a Leavitt path algebra of a finite graph and to determine the structure of its graded completion. In this way we give a precise meaning to certain infinite sums (see Sect. 3) that naturally arise in our forthcoming paper ([3]) on centers of Leavitt path algebras. Besides, the graded completion has a simpler structure than the Leavitt path algebra itself: the graded completion is semisimple , i.e. a sum of minimal graded ideals (Theorem 1).

In [3] we use the computations with infinite sums to produce descriptions of the center of the graded completion and the center of the Leavitt path algebra itself.

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Let $\Gamma = (V, E, s, r)$ be a directed graph, that consists of two sets V and E, called vertices and edges respectively and two maps $s, r : E \to V$. The vertices $s(e)$ and $r(e)$ are referred to as the source and range of the edge e respectively.

A vertex v such that $s^{-1}(v) = \emptyset$ is called a sink. A path $p = e_1 \dots e_n$ in a graph Γ is a sequence of edges $e_1, \dots, e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. We will refer to n as the length of the path p, $l(p) = n$. The edge e_n is referred to as the last edge of the path p. Vertices are viewed as paths of length 0. We say that the path p starts at the source $s(p) = s(e_1)$ and ends at the range $r(p) = r(e_n)$. The set of all paths of the graph Γ is denoted as $Path(\Gamma)$. Remark that $Path(\Gamma)$ includes all paths of length 0, that is, all vertices. If $s(p) = r(p)$ then we say that the path p is closed. If $p = e_1 \dots e_n$ is a closed path of length ≥ 1 and the vertices $s(e_1), \dots, s(e_n)$ are distinct then we call the path p a cycle. Denote $V(p) = \{s(e_1), \dots, s(e_n)\}$, $E(p) = \{e_1, \dots, e_n\}$. An edge $e \in E$ is called an exit of a cycle C if $s(e) \in V(C)$, but $e \notin E(C)$.

If X, Y are nonempty subsets of the set V then we denote $E(X, Y) = \{e \in E \mid s(e) \in X, r(e) \in Y\}$, $Path(X, Y) = \{p \in Path(\Gamma) \mid s(p) \in X, r(p) \in Y\}$.

A vertex $w \in V$ is called a descendant of a vertex $v \in V$ if $Path(\{v\}, \{w\}) \neq \emptyset$.

A nonempty subset $W \subseteq V$ is said to be hereditary if for an arbitrary element $w \in W$ all descendants of w lie in W (see [1]).

Let F be a field. The Leavitt path algebra $L(\Gamma)$ is the F-algebra presented by the sets of generators $\{v \mid v \in V\}$, $\{e, e^* \mid e \in E\}$ and the set of relations (1) $v_i v_j = \delta_{ij} v_i$ for all $v_i, v_j \in V$; (2) $s(e)e = er(e) = e$, $r(e)e^* = e^* s(e) = e^*$ for all $e \in E$; (3) $e^* f = \delta_{e,f} r(e)$ for all $e, f \in E$; (4) $v = \sum_{s(e)=v} ee^*$ for an arbitrary vertex v which is not a sink, ([1,2,4]).

The mapping $*$ which sends v to v for $v \in V$, e to e^* and e^* to e for $e \in E$, extends to an involution of the algebra $L(\Gamma)$.

The algebra $L(\Gamma)$ is $\mathbb{Z}-$graded: $\deg(v) = 0$ for all $v \in V$, $\deg(e) = 1$, and $\deg(e^*) = -1$ for all $e \in E$.

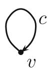
Throughout the paper we consider only finite graphs.

2 Topology on $L(\Gamma)$

We call a mapping $\gamma : V \backslash \{sinks\} \to E$ a *specialization* if $s(\gamma(v)) = v$ for an arbitrary vertex $v \in V \backslash \{sinks\}$. Edges lying in the image $\gamma(V \backslash \{sinks\})$ are called special. For a specialization γ consider the set $B(\gamma)$ of the products pq^*, where $p, q \in Path(\Gamma)$ and the following restriction holds: if the last edge of the path p is equal to the last edge of the path q then this edge is not special. Remark that since the paths p, q may have zero length all elements p, q^* and all vertices are automatically included in $B(\gamma)$.

In [5] we proved that $B(\gamma)$ is a basis of the algebra $L(\Gamma)$.

Throughout the paper we will illustrate our concepts by the following examples.

Example 1 Let $\Gamma =$ 

be a loop. The only edge c is special, thus $\gamma(v) = c$. The basis $B(\gamma)$ consist of the elements $v, c^i, (c^*)^i;\ i \geq 1$.

Example 2 Let $\Gamma = \begin{tikzcd} v \arrow[loop above, "c"] \arrow[d, "e"] \\ w \end{tikzcd}$. The Leavitt path algebra $L(\Gamma)$ is the so called algebraic Toeplitz algebra. It is isomorphic to the Jacobson algebra [9]. Assume $c^0 = v$. We have $Path(\Gamma) = \{c^i, e, c^i e, w; i \geq 0\}$. Let e be the special edge. Then $B(\gamma) = \{c^i, (c^*)^i, c^i(c^*)^j, c^i e, e^*(c^*)^j, w; i, j \geq 0\}$.

Definition 1 We call a path $p = e_1 \dots e_n$ of length $n \geq 1$ *special* if all edges $e_1, \dots, e_n$ are special.

Definition 2 For an arbitrary path $p = e_1 \dots e_n$ let i be the minimal integer such that the path $e_{i+1} \dots e_n$ is special. If the edge e_n is not special then $i = n$. We call $n - i$ the *special degree* of the path p and denote $sd(p) = n - i$.

Example 3 Since the only edge c is special it follows that an arbitrary path p of length ≥ 1 is special and $l(p) = sd(p)$.

Example 4 If the edge e is special then $sd(c^i) = 0, sd(c^i e) = 1$. If the loop c is special then $sd(c^i) = i, sd(c^i e) = 0$.

Let $p, q \in Path(\Gamma)$. We say that the path p is a *beginning* of the path q and the path q is a *continuation* of the path p if there exists a path $q' \in Path(\Gamma)$ such that $q = pq'$.

Remark 1 We will often use the following straightforward fact: if $p, q \in Path(\Gamma)$ then $p^*q \neq 0$ if and only if one of the paths p, q is a continuation of the other one.

Definition 3 For nonnegative integers n, s, d, let $V_{n,s,d}$ be the subspace of $L(\Gamma)$ F-spanned by all products pq^* such that $p, q \in Path(\Gamma)$, $l(p) + l(q) \geq n, sd(p) + sd(q) \leq s$, $|\deg(pq^*)| = |l(p) - l(q)| \leq d$.

Example 5 Let the edge e be special. An element $c^i(c^*)^j$ lies in $V_{n,s,d}$ if and only if $i + j \geq n, |i - j| \leq d$. This is the region of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ bounded by the lines $i + j = n, i - j = d, i - j = -d$. An element $c^i ee^*(c^*)^j$ lies in $V_{n,s,d}$ if and only if $i + j \geq n - 2, s \geq 2, |i - j| \leq d$. This is the region of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ bounded by the lines $i + j = n - 2, i - j = d, i - j = -d$.

Lemma 1 $V_{n_1,s_1,d_1} . V_{n_2,s_2,d_2} \subseteq V_{\left[\!\left[\frac{1}{2}(n_1+n_2-d_1-d_2)\right]\!\right], s_1+s_2, d_1+d_2}$.

Proof Let $p_i, q_i \in Path(\Gamma)$, $p_i q_i^* \in V_{n_i,s_i,d_i}, i = 1, 2$. Then $l(p_i) + l(q_i) \geq n_i, |l(p_i) - l(q_i)| \leq d_i$, which implies $l(p_i) = \frac{1}{2}(l(p_i) + l(q_i) + l(p_i) - l(q_i)) \geq \frac{1}{2}(n_i - d_i)$. Similarly, $l(q_i) \geq \frac{1}{2}(n_i - d_i)$. If $p_1 q_1^* p_2 q_2^* \neq 0$ then in view of the Remark 1 there exists a path $p_2' \in Path(\Gamma)$ such that $p_2 = q_1 p_2'$ or there exists a path $q_1' \in Path(\Gamma)$ such that $q_1 = p_2 q_1'$.

We will consider only the first case $p_2 = q_1 p_2'$. The second case is treated similarly. We have $p_1 q_1^* p_2 q_2^* = p_1 q_1^* q_1 p_2' q_2^* = p_1 p_2' q_2^*$. Clearly

$$\begin{aligned} l(p_1 p_2') + l(q_2) \geq l(p_1) + l(q_2) &\geq \frac{1}{2}(n_1 - d_1) + \frac{1}{2}(n_2 - d_2) \\ &\geq \left[\!\left[\frac{1}{2}(n_1 + n_2 - d_1 - d_2)\right]\!\right]. \end{aligned}$$

For arbitrary paths $p, q \in Path(\Gamma)$, $r(p) = r(q)$, we have $sd(pq) = sd(q)$ unless the path q is special. In the latter case $sd(pq) = sd(p) + sd(q)$. Hence $sd(q) \leq sd(pq) \leq sd(p) + sd(q)$. Applying this inequality, we get

$sd(p_1 p_2') + sd(q_2) \leq sd(p_1) + sd(p_2') + sd(q_2) \leq sd(p_1) + sd(p_2) + sd(q_2) \leq s_1 + s_2$. Finally, $\deg(p_1 q_1^* p_2 q_2^*) = \deg(p_1 q_1^*) + \deg(p_2 q_2^*)$. Hence $|\deg(p_1 q_1^* p_2 q_2^*)| \leq |\deg(p_1 q_1^*) + \deg(p_2 q_2^*)| \leq d_1 + d_2$. This completes the proof of the Lemma. $\square$

Definition 4 For a positive k let $V_k = \sum\{V_{n,s,d} \mid n \geq k(s + d + 1)\}$. Clearly, $V_{k_1} \subseteq V_{k_2}$ for $k_1 \geq k_2$.

Example 6 As we have mentioned above, $l(p) = sd(p)$ for an arbitrary path p. The subset V_1 is spanned by products pq^*; $p, q \in Path(\Gamma)$, such that $l(p) + l(q) \geq sd(p) + sd(q) + \deg(pq^*) + 1$, but this is an empty set. Hence $V_1 = (0)$.

Example 7 The element $pq^* = c^i (c^*)^j$ lies in V_k and satisfies the condition $l(p) + l(q) \geq k(sd(p) + sd(q) + \deg(pq^*) + 1)$ if and only if $i + j \geq k(|i - j| + 1)$. This is the region of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ bounded by the lines $i + j = k(i - j + 1)$, $i + j = k(j - i + 1)$.

The element $pq^* = c^i ee^*(c^*)^j$ lies in V_k and satisfies the condition $l(p) + l(q) \geq k(sd(p) + sd(q) + \deg(pq^*) + 1)$ if and only if $i + j + 2 \geq k(2 + |i - j| + 1)$. This is the region of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ bounded by the lines $i + j + 2 = k(i - j + 3)$, $i + j + 2 = k(j - i + 3)$.

Lemma 2 *Let* $k \geq 3$. *Then* $V_k V_k \subseteq V_{[\![\frac{1}{2}(k-1)]\!]}$.

Proof Suppose that $n_i \geq k(s_i + d_i + 1)$, $i = 1, 2$. Then by Lemma 1

$$V_{n_1,s_1,d_1} . V_{n_2,s_2,d_2} \subseteq V_{[\![\frac{1}{2}(n_1+n_2-d_1-d_2)]\!],s_1+s_2,d_1+d_2}.$$

We have

$$\begin{aligned} \left[\!\left[\frac{1}{2}(n_1 + n_2 - d_1 - d_2)\right]\!\right] &\geq \left[\!\left[\frac{1}{2}(n_1 + n_2 - d_1 - d_2 - s_1 - s_2)\right]\!\right] \\ &> \left[\!\left[\frac{1}{2}(k - 1)(s_1 + s_2 + d_1 + d_2)\right]\!\right] \\ &\geq \left[\!\left[\frac{1}{2}(k - 1)\right]\!\right](s_1 + s_2 + d_1 + d_2). \end{aligned}$$

$\square$

Lemma 3 *For an arbitrary element* $a \in L(\Gamma)$, *and an arbitrary integer* $k \geq 1$ *there exists* $k' \geq 1$ *such that* $aV_{k'} + V_{k'}a \subseteq V_k$.

Proof Let $a = \sum_i \alpha_i p_i q_i^*$, where $\alpha_i \in F$; $p_i, q_i \in Path(\Gamma)$. Suppose that for an arbitrary summand there exists an integer $k_i \geq 1$ such that $p_i q_i^* V_{k_i} + V_{k_i} p_i q_i^* \subseteq V_k$. Then for $k' = \max_i k_i$ we have $aV_{k'} + V_{k'}a \subseteq V_k$. Thus without loss of generality we can assume that $a = pq^*$, where p, q are paths. Then $a \in V_{n_0,s_0,d_0}$, $n_0 = l(p) + l(q)$, $s_0 = sd(p) + sd(q)$, $d_0 = |\deg(a)|$.

Let $n \geq 0$, $s \geq 0$, $d \geq 0$. By Lemma 1 we have

$$V_{n_0,s_0,d_0} V_{n,s,d} \subseteq V_{\left[\!\left[\frac{1}{2}(n+n_0-d-d_0) \right]\!\right], s+s_0, d+d_0}.$$

For the right hand side to lie in V_k it is sufficient to have $\frac{1}{2}(n-(s+d)+n_0-d_0) \geq k(s+d+s_0+d_0+1)$ or, equivalently $\frac{1}{2}(n-(s+d)) \geq k(s+d)+c$, where $c = k(s_0+d_0+1) - \frac{1}{2}(n_0-d_0)$. If $n \geq k'(s+d+1)$, then $\frac{1}{2}(n-(s+d)) \geq \frac{1}{2}(k'-1)(s+d) + \frac{1}{2}k'$. Hence, for $k' \geq \max\{2k+1, 2c\}$ the inclusion of the Lemma holds. □

Lemma 4 $\bigcap_{k \geq 1} V_k = (0)$.

Proof Recall that the basis $B(\gamma)$ of the algebra $L(\Gamma)$ that corresponds to the specialization γ consists of products pq^*, where $p, q \in Path(\Gamma)$, $r(p) = r(q)$ and if the last edge of p is equal to the last edge of q then this edge is not special. Let $V_{(n)}$ denote the F-span of all products $pq^* \in B(\gamma)$ such that $l(p) + l(q) \geq n$.

Clearly, $\bigcap_{n \geq 1} V_{(n)} = (0)$. It is easy to see that $V_{n,d,s} \subseteq V_{(n-s)}$. Hence $V_k \subseteq V_{(k)}$ for $k \geq 1$, which implies the assertion of the Lemma. □

The subspaces $\{V_k\}_{k \geq 1}$ form a basis of neighborhoods of 0 in $L(\Gamma)$ and define a topology. By Lemmas 2, 3 this topology is compatible with the algebra structure. Let $\overline{L(\Gamma)}$ be the completion of the topological algebra $L(\Gamma)$. Let $\overline{L(\Gamma)_i}$ denote the completion of the homogeneous component $L(\Gamma)_i$ of degree i in the algebra $\overline{L(\Gamma)}$. The main focus of this paper will be on the completion

$$\widehat{L}(\Gamma) = \sum_{i \in \mathbb{Z}} \overline{L(\Gamma)_i}.$$

Example 8 Since $V_1 = (0)$ it follows that the topology on $L(\Gamma)$ is discrete. The graded completion $\widehat{L}(\Gamma)$ is isomorphic to the original Leavitt path algebra.

Example 9 Consider the countably dimensional vector space $X = \sum_{i=0}^{\infty} Fe_i$. Let $E(X)$ be the algebra of all linear transformation of X. Since the basis $\{e_i, i \geq 0\}$ has been fixed we can identify $E(X)$ with the algebra of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$-matrices having only finitely many nonzero entries in each column.

Consider the subalgebra $M_\infty(F) < E(X)$ of all (finitary) matrices having finitely many nonzero entries and the subalgebra $\widehat{M}_\infty(F)$ of matrices having finitely many nonzero diagonals, $\widehat{M}_\infty(F) = \{(a_{ij})_{i,j \geq 1} \mid a_{ij} \in F$, there exists $d \geq 1$ such that $a_{ij} = 0$ whenever $|i-j| > d\}$. Let $e_{i,j}; i, j \geq 0$, denote the matrix unit having 1 at the intersection of the ith row and the jth column and zeros elsewhere. In [6] we identified the algebraic Toeplitz algebra $L(\Gamma)$ with a subalgebra of the algebra $\widehat{M}_\infty(F)$: the elements c, c^* are identified with the infinite matrices $\sum_{i=0}^{\infty} e_{i+1,i}, \sum_{i=0}^{\infty} e_{i,i+1}$; the elements $w, c^i e, e^*(c^*)^i, c^i ee^*(c^*)^j$ are identified with the matrix units $e_{0,0}, e_{i+1,0}, e_{0,i+1}, e_{i+1,j+1}$ respectively.

Thus $L(\Gamma) = \langle c, c^*, M_\infty(F) \rangle < \widehat{M}_\infty(F)$.

The degree function $d(e_{i,j}) = i - j$ makes $\widehat{M}_\infty(F)$ a $\mathbb{Z}$-graded algebra, $L(\Gamma)$ is a graded subalgebra of $\widehat{M}_\infty(F)$. The homogeneous component of degree d of the algebra $\widehat{M}_\infty(F)$ is the diagonal $\{\sum_{i,j} \alpha_{i,j} e_{i,j} \mid \alpha_{i,j} \in F, i, j \geq 0, i - j = d\}$. From the description of subspaces V_k in $L(\Gamma)$ it follows that for arbitrary $k \geq 1, d \in \mathbb{Z}$ there exists a sufficiently large integer $m = m(k,d)$ such that all finitary matrices $\sum_{i,j} \alpha_{i,j} e_{i,j} \in M_\infty(F), i, j \geq m, i - j = d$, lie in V_k. Hence the component of degree d of the algebra $M_\infty(F)$ is dense in the component of degree d of the algebra $\widehat{M}_\infty(F)$. This implies $\widehat{L}(\Gamma) = \sum_{d \in \mathbb{Z}} \overline{L(\Gamma)_d} = \widehat{M}_\infty(F)$.

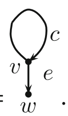
Lemma 5 *Let $k \geq 2$. Let a be an element from V_k and let $a = \sum_i \mu_i a_i$ be the presentation of a as a linear combination of basic elements $a_i \in B(\gamma)$, $\mu_i \in F$. Then all the elements a_i lie in V_{k-1}.*

Proof Without loss of generality we will assume that $a = pq^*$; $p, q \in Path(\Gamma)$, $r(p) = r(q)$, $l(p) + l(q) \geq k(sd(p) + sd(q) + \deg(pq^*) + 1)$. The presentation $a = \sum_i \mu_i a_i$ is obtained by the Groebner-Shirshov algorithm (see [5,8]). All the basic elements a_i are of the types $a_i = p_i q_i^*$, where $p_i, q_i \in Path(\Gamma)$, $l(p_i) = l(p) - r$, $l(q_i) = l(q) - r$, $\deg(p_i q_i^*) = \deg(pq^*)$. Now, $l(p_i) + l(q_i) = l(p) + l(q) - 2r \geq k(sd(p_i) + sd(q_i) + 2r + \deg(p_i q_i^*) + 1) - 2r \geq (k-1)(sd(p_i) + sd(q_i) + \deg(p_i q_i^*) + 1)$, which finishes the proof of the Lemma. □

Lemma 6 *$B(\gamma)$ is a topological basis of the algebra $\widehat{L}(\Gamma)$, i.e. an arbitrary element of $\widehat{L}(\Gamma)$ can be uniquely represented as a converging series $\sum_{b \in B(\gamma)} \alpha_b b$, $\alpha_b \in F$.*

Proof An arbitrary element of $\widehat{L}(\Gamma)$ can be represented as a converging sum $\sum_{i \in \Omega} a_i$, where $\{a_i \in L(\Gamma), i \in \Omega\}$ is a Cauchy set. In other words for an arbitrary $k \geq 1$ the set $\{i \in \Omega \mid a_i \notin V_k\}$ is finite. Let $a_i = \sum_j \alpha_{ij} a_{ij}$, $1 \leq j \leq t_i$, be the decomposition of a_i as a linear combination of distinct basic elements from $B(\gamma)$, $\alpha_{ij} \neq 0$, $a_{ij} \in B(\gamma)$. From Lemma 5 it follows that $\{a_{ij}, i \in \Omega, 1 \leq j \leq t_i\}$ is also a Cauchy set. Hence $\sum a_i = \sum \alpha_{ij} a_{ij}$.

Let $\sum \alpha_b b = 0$, $\alpha_b \in F$, $b \in B(\gamma)$. Suppose that $\alpha_{b_0} \neq 0 \neq$ and $b_0 \notin V_k$. Then by Lemma 5 any finite subsum of $\sum \alpha_b b$, containing $\alpha_{b_0} b_0$, does not belong to V_{k+1}, a contradiction. This completes the proof of Lemma. □

Example 10 Consider $\Gamma =$ 

. Let $\gamma(v) = c$. We have shown above that the graded completion $\widehat{L}(\Gamma)$ can be identified with the algebra $\widehat{M}_\infty(F)$ of infinite matrices having finitely many nonzero diagonals. The basis $B(\gamma)$ in this case consists of matrix units. Clearly, it is a topological basis in $\widehat{M}_\infty(F)$.

We will need another general statement about sums in $\widehat{L}(\Gamma)$. Consider a nonzero converging sum $a = \sum_{i \in \Omega} a_i \in \widehat{L}(\Gamma)$, $a_i \in L(\Gamma)$. We say that the sum is *reduced* if for any arbitrary nonempty subset $\Omega' \subset \Omega$ we have $\sum_{i \in \Omega'} a_i \neq 0$.

Lemma 7 *For an arbitrary nonzero converging sum $a = \sum_{i \in \Omega} a_i$, $a_i \in L(\Gamma)$, there exists a nonempty subset $\Omega' \subset \Omega$ such that $a = \sum_{i \in \Omega'} a_i$ and this sum is reduced.*

Proof Let $\Omega_1 \subset \Omega_2 \subset \cdots$ be an ascending chain of subsets of Ω such that $\sum_{i\in\Omega_k} a_i = 0$ for each k. Denote $\widetilde{\Omega} = \cup_k \Omega_k$.

We claim that $\sum_{i\in\Omega} a_i = 0$. Indeed, since the sum $\sum_{i\in\Omega} a_i$ is convergent it follows that for an arbitrary $t \geq 1$ the set $\{i \in \Omega \mid a_i \notin V_t\}$ is finite. Hence there exists $k \geq 1$ such that $a_i \in V_t$ for any $i \in \widetilde{\Omega}\backslash\Omega_k$.

Now, $\sum_{i\in\widetilde{\Omega}} a_i = \sum_{i\in\Omega_k} a_i + \sum_{i\in\widetilde{\Omega}\backslash\Omega_k} a_i \in \overline{V_t}$. This implies that $\sum_{i\in\widetilde{\Omega}} a_i \in \cap_{t\geq 1}\overline{V_t} = (0)$. By Zorn's Lemma there exists a maximal subset $\Omega_{max} \subset \Omega$ such that $\sum_{i\in\Omega_{max}} a_i = 0$. Let $\Omega' = \Omega\backslash\Omega_{max}$. Then $a = \sum_{i\in\Omega'} a_i$ and this sum is reduced. □

3 Central idempotents in $\widehat{L}(\Gamma)$

Lemma 8 *Let $p = e_1 \ldots e_n$ be a special path and $r(e_n) \notin \{s(e_1), \ldots, s(e_n)\}$. Then $n \leq |V|$.*

Proof If $n > |V|$ then some vertex on p appears at least twice and this vertex is not $r(e_n)$. Hence, a subpath p_1 of p is a cycle. Since $r(e_n)$ does not lie in $V(p_1)$ it follows that some exit from the cycle p_1 is special. But this is impossible since for every non-sink vertex v only one edge from $s^{-1}(v)$ is special. □

Definition 5 Let $W \subset V$ be a nonempty subset. We say that a path $p = e_1 \ldots e_n$, $e_i \in E$, is an *arrival path* in W if $r(p) \in W$, and $\{s(e_1), \ldots, s(e_n)\} \nsubseteq W$. In other words, $r(p)$ is the first vertex on p that lies in W. In particular, every vertex $w \in W$, viewed as a path of zero length, is an arrival path in W. Let $Arr(W)$ be the set of all arrival paths in W.

Lemma 9 *The set $\{pp^* \mid p \in Arr(W)\}$ is a Cauchy set.*

Proof We need to check that for an arbitrary $k \geq 1$ the set $\{pp^* \mid p \in Arr(W)\}\backslash V_k$ is finite. If p is an arrival path in W, then by Lemma 8 $sd(pp^*) \leq 2|V|$, $d(pp^*) = 0$. Hence $\{pp^* \mid p \in Arr(W)\}\backslash V_k \subseteq \{pp^* \mid l(p) < k(|V| + \frac{1}{2})\}$. Clearly, it is a finite set, which completes the proof. □

Denote $e(W) = \sum\limits_{p\in Arr(W)} pp^* \in \widehat{L}(\Gamma)$.

If $W = \phi$ then we let $e(W) = 0$.

Example 11 The only proper hereditary subset of V is $W = \{w\}$. We have $Arr(W) = \{w, c^i e \mid i \geq 0\}$,

$$\begin{aligned} e(W) &= w + \sum_{i\geq 0} c^i ee^*(c^*)^i \\ &= w + \sum_{i\geq 0} c^i(v - cc^*)(c^*)^i \\ &= w + \sum_{i\geq 0} c^i(c^*)^i - \sum_{i\geq 1} c^i(c^*)^i \\ &= w + v = 1. \end{aligned}$$

Lemma 10 *If W is a hereditary set, then $e(W)$ is a central idempotent in $\widehat{L}(\Gamma)$.*

Proof If a, b are distinct elements from $\{pp^* \mid p \in Arr(W)\}$ then $ab = ba$ by Remark 1. Hence, $e(W)$ is a sum (possibly infinite) of pairwise orthogonal idempotents. Hence $e(W)$ is an idempotent. Since $L(\Gamma)$ is dense in $\widehat{L}(\Gamma)$ it is sufficient to show that $e(W)$ commutes with all vertices and all edges of Γ. For a vertex $v \in V$ let $Arr(v, W) = \{p \in Arr(W) \mid s(p) = v\}$. If $w \in W$ then $Arr(w, W) = \{w\}$. It is easy to see that $v.e(W) = e(W).v = \sum_{p \in Arr(v,W)} pp^*$. Let $e \in E$. We will consider 3 cases:

Case 1. $r(e) \notin W$. Then $e \sum_{p \in Arr(W)} pp^* = e \sum_{p \in Arr(r(e),W)} pp^*$; $\sum_{p \in Arr(W)} pp^* e = \sum\{pp^* e \mid p \in Arr(W),$ the first edge of p is $e\}$. It is easy to see that these two sums are equal.

Case 2. $r(e) \in W$, $s(e) \notin W$. Then $e \in Arr(W)$. We have $e \sum_{p \in Arr(W)} pp^* = er(e)r(e)^* = e$; $\sum_{p \in Arr(W)} pp^* e = ee^* e = e$.

Case 3. $r(e) \in W$, $s(e) \in W$. In this case we again have $e \sum_{p \in Arr(W)} pp^* = e$; $\sum_{p \in Arr(W)} pp^* e = s(e)s(e)^* e = e$. □

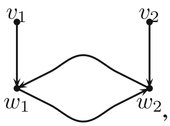
Remark 2 In [3] we will show that all central idempotents of the algebra $\widehat{L}(\Gamma)$ are of the type $e(W)$, where W is a nonempty hereditary subset of V.

4 Frames

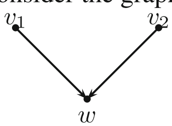
Let W a nonempty subset of V. We will define a graph $\Gamma^W = (V', E')$ as follows:

$V' = (V \backslash W) \cup \{w\}$, where w is a new vertex, not belonging to V; for two vertices $v_1, v_2 \in V \backslash W$, the set of edges $E'(\{v_1\}, \{v_2\})$ is identified with $E(\{v_1\}, \{v_2\})$; the set of edges $E'(\{v_1\}, \{w\})$ is identified with $E(\{v_1\}, W)$. For an edge $e \in E(\{v_1\}, W)$ and its image e' in $E'(\{v_1\}, \{w\})$ we will say that e' is the edge e redirected to w.

Remark 3 Since all edges $E(W, V)$ are ignored, the vertex w is a sink in Γ^W.

Example 12 Consider the graph $\Gamma =$ 

, $W = \{w_1, w_2\}$.

Then $\Gamma^W =$ 

Lemma 11 *Let $\Gamma = (V, E)$ be a finite graph with a sink v which is a descendant of every vertex in V. Then there exists a specialization $\gamma : V \backslash \{v\} \to E$, such that the set of all special paths in Γ is finite.*

Proof Let $v_1, \ldots, v_k \in V$ be vertices such that $E(\{v_i\}, \{v\}) \neq \emptyset$. In each set $E(\{v_i\}, \{v\})$ choose one edge and declare it special. All other edges coming out of

$v_1, \ldots, v_k$ are not special. Consider the graph $\Gamma' = \Gamma^{\{v_1,\ldots,v_k,v\}} = (V', E')$, $V' = (V \backslash \{v_1, \ldots, v_k\}) \cup \{w\}$. Since v is a descendant of an arbitrary vertex in V it follows that v is the only sink in Γ. Similarly, w is a descendant of an arbitrary vertex in V', hence, w is the only sink in Γ'. Since $|V'| < |V|$ by the induction assumption there exists a specialization $\gamma' : V' \backslash \{w\} \to E'$ such that the set of special paths in Γ' is finite. Now we are ready to construct the specialization $\gamma : V \backslash \{v\} \to E$. Choose a vertex $u \in V \backslash \{v_1, \ldots, v_k, v\}$ and let $\gamma'(u) = e' \in E'$. If $r(e') \neq w$ then $e' \in E(V \backslash \{v_1, \ldots, v_k, v\}, V \backslash \{v_1, \ldots, v_k, v\})$ and we define $\gamma(u) = e'$. Now let $r(e') = w$. It means that there was an edge $u \xrightarrow{e} v_i$, $1 \le i \le k$, that was redirected to $u \xrightarrow{e'} w$. We let $\gamma(u) = e$. If $u \in \{v_1, \ldots, v_k\}$ then at the beginning of the proof we chose a special edge $u \to v$. We claim that with the specialization γ defined above there are finitely many special paths in Γ.

If not, then Γ contains a special cycle. This cycle can not involve any of the vertices $v_1, \ldots, v_k$, since special edges from $v_1, \ldots, v_k$ lead to v, a sink. Hence this cycle lies in Γ', which contradicts the induction assumption. This proves the Lemma. □

Let W be a minimal nonempty hereditary subset of V. Then for any two vertices $w_1, w_2 \in W$ the vertex w_2 is a descendant of w_1. Indeed, the set of all descendants of w_1 is a hereditary subset of V. In view of minimality of W it contains W. It implies that for any two minimal hereditary subset W_1, W_2 either $W_1 = W_2$ or $W_1 \cap W_2 = \emptyset$.

Definition 6 The collection $W_1, \ldots, W_k$ of all distinct minimal nonempty hereditary subsets of V is called the frame of Γ.

Lemma 12 *Every vertex of* Γ *has a descendant in* $\cup_{i\ge1}^k W_i$.

Proof The set of all vertices that do not have a descendant in $\cup_{i\ge1}^k W_i$ is hereditary. If nonempty, then it contains one of the subsets $W_1, \ldots, W_k$, a contradiction. This proves the Lemma. □

Definition 7 We say that a specialization $\gamma : V \backslash \{sinks\} \to E$ is a frame specialization if the set of all special paths $p = e_1 \ldots e_n$ with $s(e_1), \ldots, s(e_n) \notin \bigcup\limits_{i\ge1}^{k} W_i$ is finite.

Lemma 13 *An arbitrary finite graph* Γ *has a frame specialization.*

Proof Consider the graph $\Gamma' = \Gamma^{W_1 \cup \ldots \cup W_k} = (V', E')$, $V' = (V \backslash (\cup_{i\ge1}^k W_i)) \cup \{w\}$. This graph contains a sink w, which is a descendant of all vertices in V' by Lemma 12. By Lemma 11 there exist a specialization $\gamma' : V' \backslash \{w\} \to E'$ such that the set of all special paths in Γ' is finite.

Let's define a specialization $\gamma : V \backslash \{sinks\} \to E$. For non-sinks from $\cup_{i\ge1}^k W_i$ define γ arbitrarily. Choose a vertex $u \in V \backslash (\cup_{i\ge1}^k W_i)$. Clearly, u is not a sink in Γ'. If $r(\gamma'(u)) \neq w$ then we let $\gamma(u) = \gamma'(u)$. If $r(\gamma'(u)) = w$ then $\gamma'(u)$ has been redirected from some edge $e \in E$, $s(e) = u$, $r(e) \in \cup_{i\ge1}^k W_i$. Let $\gamma(u) = e$. If $p = e_1 \ldots e_n$ is a special path in Γ such that $s(e_1), \ldots, s(e_n) \in V \backslash (\cup_{i\ge1}^k W_i)$ then p can be viewed as a special path in Γ'. Since there are finitely many such paths, this completes the proof of the Lemma. □

Example 13 The only minimal nonempty hereditary subset is $\{w\}$. If the cycle c is special then there are infinitely many special path c^i, $i \geq 1$, having all vertices outside of $\cup_{i\geq 1}^k W_i = \{w\}$. If the edge e is special then there is only one such special path e.

From now on we will consider only frame specialization.

Lemma 14 *Let $W' \subset W'' \subset V$ be hereditary subsets such that every vertex from W'' has a descendant in W'. Then $e(W') = e(W'')$.*

Proof Let $W_1, \ldots, W_k$ be the frame of the graph Γ. Let $\gamma : V\backslash\{sinks\} \to E$ be a specialization that satisfies the condition of Lemma 13.

Since every vertex in W'' has a descendant in W' it follows that for an arbitrary minimal hereditary subset W_i either $W_i \cap W'' = \phi$ or $W_i \subseteq W'$. Choose a vertex $v \in W''\backslash W'$. As above we denote $Arr(v, W') = \{p \in Arr(W') \mid s(p) = v.\}$ We claim that $v = \sum_{p\in Arr(v,W')} pp^*$. To prove this equality we will define a sequence of finite sets of paths $P_0, P_1, \ldots$. Let $P_0 = \{v\}$. If P_n has been constructed then P_{n+1} is defined in the following way. Let $p \in P_n$. If $r(p) \in W'$ then $p \in P_{n+1}$. Let $r(p) \in W''\backslash W'$. Then $r(p)$ is not a sink (the set $W''\backslash W'$ does not contain sinks). Let $e_1, \ldots, e_q$ be all edges with the source at $r(p)$. Then $pe_1, \ldots, pe_q \in P_{n+1}$. Thus $P_{n+1} = \{p \in P_n, r(p) \in W'\}\dot{\cup}\{pe \mid p \in P_n, r(p) \in W''\backslash W', s(e) = r(p)\}$. For an arbitrary $n \geq 0$ we have $v = \sum_{p\in P_n} pp^*$. If $p \in P_n$ and $r(p) \in W'$ then p is an arrival path in W'. Since γ is a frame specialization it follows that all $sd(p)$, $p \in \cup_{n\geq 0} P_n$, are uniformly bounded from above. Hence $\sum_{p\in P_n\backslash Arr(W')} pp^* \to 0$. It follows that

$$v = \lim_{n\to\infty} \sum_{p\in P_n\cap Arr(W')} pp^* = \sum_{p\in Arr(v,W')} pp^*.$$

Now,

$$e(W') = \sum_{p\in Arr(W'')} p \left(\sum_{p_1\in Arr(r(p),W')} p_1 p_1^* \right) p^* = \sum_{p\in Arr(W'')} pp^* = e(W'').$$

□

Let W be a nonempty hereditary subset of V. Let $W^\perp \subset V$ consist of those vertices which do not have descendants in W. Clearly, $W^\perp$ is a hereditary subset of V.

Lemma 15 *The idempotents $e(W), e(W^\perp)$ are orthogonal and $e(W) + e(W^\perp) = 1$.*

Proof If p, q are arrival paths to $W, W^\perp$ respectively then none of them is a continuation of the other one. Hence $p^*q = q^*p = 0$. It implies that $e(W)e(W^\perp) = e(W^\perp)e(W) = 0$.
An arbitrary vertex from V has a descendant in $W \cup W^\perp$. Indeed, if $v \in V$ and v does not have descendants in W then $v \in W^\perp$. By Lemma 14, $e(W) + e(W^\perp) = e(W \cup W^\perp) = e(V) = 1$. This finishes the proof of the Lemma. □

Corollary 1 $e(W) = e((W^\perp)^\perp)$.

Lemma 16 $(W^{\perp})^{\perp}$ *is the largest hereditary subset of* V *such that every vertex of it has a descendant in* W.

Proof Since $(W^{\perp})^{\perp} \cap W^{\perp} = \phi$ we conclude that every vertex from $(W^{\perp})^{\perp}$ has a descendent in W. Now let $U \subseteq V$ be a nonempty hereditary subset such that every vertex from U has a descendant in W. In order to prove $V \subseteq (W^{\perp})^{\perp}$ we need to show that no vertex $u \in U$ can have a descendant in $W^{\perp}$. Let v be a descendant of the vertex u that lies in $W^{\perp}$. Since U is hereditary it follows that $v \in U$. Hence, v has a descendant in W. It contradicts the inclusion $v \in W^{\perp}$ and completes the proof. □

The closed ideal of the algebra $\widehat{L}(\Gamma)$ generated by the hereditary subset $W \subset V$ consists of (possibly infinite) converging sums $\sum \alpha_{pq} pq^*$, where $\alpha_{pq} \in F$; $p, q \in Path(\Gamma)$, $r(p) = r(q) \in W$. We will denote this ideal as $I(W)$.

Lemma 17 $I(W) = e(W)\widehat{L}(\Gamma)e(W)$.

Proof For an arbitrary vertex $w \in W$ we have $w = we(W)$. Hence $W \subset e(W)\widehat{L}(\Gamma)$ and so $I(W) \subseteq e(W)\widehat{L}(\Gamma)$. The inclusion $e(W) \in I(W)$ follows from the fact that every arrival path in W ends with a vertex from W. This finishes the proof of the Lemma. □

Lemma 17 implies that the ideal $I(W)$ is a direct summand of the algebra $\widehat{L}(\Gamma) = I(W) \oplus I(W^{\perp})$ and that $\widehat{L}(\Gamma) = I(W_1) \oplus \cdots \oplus I(W_k)$. Now our aim is to decompose $\widehat{L}(\Gamma)$ as a direct sum of minimal ideals.

5 Completions of simple Leavitt path algebras and the ideals $I(W_i)$

Recall that $\gamma : V \to E$ is a fixed frame specialization of the algebra Γ. If W is a hereditary subset of V then $\gamma(W) \subseteq E(W, W)$.

For a vertex $v \in V$ define a special path $g_v(n)$ inductively. Let $g_v(0) = v$. If $r(g_v(n))$ is not a sink then $g_v(n+1) = g_v(n)\gamma(r(g_v(n)))$. If $r(g_v(u))$ is a sink then $g_v(n+1) = g_v(n)$.

Remark that an arbitrary $n \geq 0$ the element $g(n)g(n)^*$ is an idempotent.

Lemma 18 $\{g_v(n)g_v(n)^*, n \geq 0\}$ *is a Cauchy set.*

Proof For a vertex $v \in V$ let $\mathcal{E}(v)$ denote the set of all non special edges e with $s(e) = v$. Then

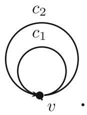
$$g_v(n+1)g_v(n+1)^* = g_v(n)g_v(n)^* - \sum_{e \in \mathcal{E}(r(g_v(n)))} g_v(n)ee^*g_v(n)^*,$$

which implies $g_v(n+1)g_v(n+1)^* - g_v(n)g_v(n)^* \in V_{2(n+1)}$. □

Let

$$e_v = \lim_{n \to \infty} g_v(n)g_v(n)^*.$$

Remark that e_v is an element of the graded closure $\widehat{L}(\Gamma)$.

Example 14 Consider the Leavitt graph 

. Let the edge c_1 be special. Then $g_v(n) = c_1^n$ and therefore $e_v = \lim_{n\to\infty} c_1^n(c_1^*)^n = v - \sum_{n\geq 0} c_1^n c_2 c_2^*(c_1^*)^n$.

Lemma 19 *Let $\gamma : V\backslash\{sinks\} \to E$ be a frame specialization, $v \in V$.*
Then
(1) *e_v is a non zero idempotent* , (2) *if $v \neq w$ then e_v, e_w are orthogonal* , (3) *if e is a non special edge from $E(v, V)$ then $e_v e = 0$; if $e = \gamma(v)$ then $e_v e = e e_w$, where $w = r(e)$.*

Proof Since for an arbitrary $n \geq 1$ the element $g_v(n)g_v(n)^*$ is an idempotent it follows that the limit e_v is an idempotent as well. Let us show that $e_v \neq 0$. Indeed, denote $v_n = r(g_v(n))$, $v_0 = v$. Then

$$g_v(n)g_v(n)^* = v - \sum_{e\in\mathcal{E}(v_i)} g_v(i)ee^*g_v(i)^*.$$

Hence, $e_v = v - \sum_{k\geq 1} a_k$, where $a_k = g_v(k)(\sum_{e\in\mathcal{E}(v_k)} ee^*)g_v(k)^* \to 0$ as $k \to \infty$. Since $B(\gamma)$ is a topological basis of $\widehat{L}(\Gamma)$ by Lemma 6, it implies that $e_v \neq 0$.

For distinct vertices $v, w \in V$ we have $e_v e_w = 0$ since $e_v \in v\widehat{L}(\Gamma)v$.

If $e \in E(v, V)$ and e is not special then $e_v e = 0$ since $g_v(n)^*e = 0$ for $n \geq 1$. If $e = \gamma(v)$ then $g_v(n)^*e = g_w(n-1)^*$, where $w = r(e)$. Hence, $g_v(n)g_v(n)^*e = eg_w(n-1)g_w(n-1)^*$. This finishes the proof of the Lemma. □

Consider the graph $(V, \gamma(V))$ with the set of vertices V and the set of edges $\gamma(V)$. Let $\tilde{\gamma}(V)$ be the set $\gamma(V)$ with all edges having lost their directions, $(V, \tilde{\gamma}(V))$ is the corresponding undirected graph.

Definition 8 A vertex w is a special descendant of a vertex v if there exists a special path p in Γ such that $s(p) = v$, $r(p) = w$.

Lemma 20 *Vertices $v, w \in V$ are connected in $(V, \tilde{\gamma}(V))$ if and only if they have a common special descendant in Γ.*

Proof Let $p = \tilde{e}_1 \ldots \tilde{e}_n$ be a path in $(V, \tilde{\gamma}(V))$ that connects v and w such that the length $l(p) = n$ is minimal. The (undirected) edge $\tilde{e}_1$ connects $v_1 = v$ with a vertex v_2, the edge $\tilde{e}_2$ connects v_2 with v_3, and so on. All the vertices $v_1 = v, v_2, \ldots, v_{n+1} = w$ are distinct. If $v_1 \to v_2$, then v_2 and w have a common special descendant in Γ as the distance between them in $(V, \tilde{\gamma}(V))$ is $n - 1$. Hence v_1, w have a common special descendant.

Let $v_1 \leftarrow v_2$. Since there is a unique special edge in Γ with the source v_2 it follows that $v_2 \leftarrow v_3$ and similarly $v_3 \leftarrow v_3 \leftarrow \cdots v_n \leftarrow w$. Now v is a special descendant of w which finishes the proof of the Lemma. □

Lemma 21 (1) *If vertices $v, w \in V$ are connected in $(V, \tilde{\gamma}(V))$ then e_v, e_w generate the same closed ideal in $\widehat{L}(\Gamma)$;*
(2) *If $v, w \in V$ are not connected in $(V, \tilde{\gamma}(V))$ then $e_v\widehat{L}(\Gamma)e_w = (0)$.*

Proof Let vertices $v, w \in V$ be connected in $(V, \tilde{\gamma}(V))$. Without loss of generality we can assume that there is a special edge $e \in \gamma(V)$ such that $v \to w$ *or* $v \leftarrow w$. In the first case $e_w = e^* e_v e$ by Lemma 19(3). In the second case $e_w = e e_v e^*$. In both cases e_w lies in the ideal generated e_v, which proves the claim (1).

Now let v, w lie in different connected components of $((V, \tilde{\gamma}(V))$. Since $L(\Gamma)$ is dense in $\widehat{L}(\Gamma)$ it is sufficient to prove that $e_v L(\Gamma) e_w = (0)$. Let $p, q \in Path(\Gamma), r(p) = r(q), s(p) = v, s(q) = w$. We need to show that $e_v pq^* e_w = 0$. If p is not a special path then $e_v p = 0$ by Lemma 19(3) and similarly $q^* e_w = 0$ if the path q is not special. If both paths p, q are special then Lemma 19(3) implies $e_v p = p e_{r(p)}, q^* e_w = e_{r(q)} q^*, r(p) \neq r(q)$. Hence v, w do not have a common special descendant. Hence $e_v pq^* e_w = p e_{r(p)} e_{r(q)} q^* = 0$, which finishes the proof. □

Lemma 22 *Let I be a non zero closed graded ideal of $\widehat{L}(\Gamma)$. Then $I_0 = I \cap \widehat{L}(\Gamma)_0 \neq (0)$.*

Proof Choose a nonzero homogenous element $a \in I$. Without loss of generality we can assume that there exist vertices $v, w \in V$ such that $a = vaw$. If $deg(a) = 0$ then we are done. Suppose that $\deg(a) = d \geq 1$.

The vertex v can be represented as $v = \sum_i p_i p_i^*$, where $p_i \in Path(\Gamma)$ and for an arbitrary i either $l(p_i) = d$ or $l(p_i) < d$ and $r(p_i)$ is a sink.

Let $a = \sum \alpha_{p,q} pq^*; \alpha_{p,q} \in F; p, q \in Path(\Gamma); r(p) = r(q); deg(pq^*) = d$ for every p, q. From $\deg(pq^*) = d \geq 1$ it follows that $l(p) = d + l(q) \geq d$ for each summand d.

Suppose that $l(p_i) < d, r(p_i)$ is a sink and nevertheless $p_i p_i^* pq^* \neq 0$. The path p can not be a continuation of the path p_i since $l(p_i) < l(p)$ and $r(p_i)$ is a sink. The path p_i can not be continuation of path p since $l(p) \geq d > l(p_i)$, a contradiction.

Hence , for all $p_i p_i^*$ such that $l(p_i) < d$, we have $p_i p_i^* a = 0$. This implies a $a \in \widehat{L}(\Gamma_d)\widehat{L}(\Gamma_{-d})a \subseteq \widehat{L}(\Gamma_d) I_0$. The case $\deg(a) \leq -1$ is treated similarly. □

Proposition 1 *Let W be a nonempty hereditary subset of V and let J be a nonzero closed graded ideal of $\widehat{L}(\Gamma)$ such that $J \subseteq I(W)$. Then there exists a vertex $w \in W$ such that $e_w \in J$.*

Proof By Lemma 22 the space J_0 contains a nonzero element $a = \sum \alpha_{p,q} pq^*, l(p) = l(q), r(p) = r(q) \in w$. By Lemma 7 we can assume that the sum is reduced. Denote $\mathcal{P} = \{(p, q) \in Path(\Gamma) \times Path(\Gamma) \mid \alpha_{p,q} \neq 0\}$. Choose $(p_0, q_0) \in \mathcal{P}$ with minimal length $l(p_0)$. Let $r(p_0) = v \in W$.

Let $\mathcal{P}(p_0, q_0) = \{(p, q) \in \mathcal{P} \mid p$ and $q$are proper continuations of paths p_0, q_0 respectively $\}$, $\mathcal{P}'(p_0, q_0) = \{(p, q) \in Path(\Gamma) \times Path(\Gamma) \mid (p_0 p, q_0 q) \in \mathcal{P}(p_0, q_0)\}$.

Then $a' = p_0^* a q_0 = \alpha_{p_0,q_0} v + \sum_{(p,q) \in \mathcal{P}'(p_0,q_0)} \alpha_{p,q} pq^*$ and $p_0 a' q_0^* = \alpha_{p_0,q_0} p_0 q_0^* + \sum_{(p,q) \in \mathcal{P}(p_0,q_0)} \alpha_{p,q} pq^* \neq 0$, since the sum is reduced. Hence, $a' \neq 0$.

Remark, that $a' = va'v$.

Rewriting each summand $pq^*, (p, q) \in \mathcal{P}'(p_0, q_0)$, as a linear combination of basic elements from $B(\gamma)$ and using Lemmas 5, 6 we get $a' = \sum \beta_{p,q} pq^*$, where $\beta_{p,q} \in F$, $l(p) = l(q), s(p) = s(q) = v, pq^* \in B(\gamma)$ for each summand and the sum is reduced. Remark that since the subset W is hereditary it follows that $r(p) = r(q) \in W$ for each summand. As we did before choose a summand $p_0'q_0'^*$ with minimal $l(p_0')$. If $p = p_0'p', q = q_0'q'$ and $pq^* \in B(\gamma)$ then $p'q'^* \in B(\gamma)$ as well. Now, $b = (\beta_{p_0'p'})^{-1} p_0'^* a' q_0' = w + \sum \mu_{p,q} pq^*$ is a nonzero element from $J_0, s(p) = s(q) = w, l(p) = l(q) \geq 1$, $pq^* \in B(\gamma)$ for each summand.

An infinite series $\sum_{n \geq 1} a_n$, $a_n \in L(\Gamma)$, converges in $\widehat{L}(\Gamma)$ if and only if for an arbitrary $k \geq 1$ all but finitely many elements a_n lie in V_k.

Since the sum $\sum \mu_{p,q} pq^*$ is convergent it follows that the set $\{(p, q) \in Path(\Gamma) \times Path(\Gamma) \mid \mu_{p,q} \neq 0, pq^* \notin V_2\}$ is finite.

If $pq^* \in V_2$ then $2l(p) \geq 2(sd(p) + sd(q) + 1)$, which implies that both paths p, q are not special.

For an arbitrary basic element $t \in B(\gamma)$, of degree 0 which is not a vertex, we have $g_w(n)^* t g_w(n) = 0$ for a sufficiently large n. Indeed, let $t = pq^*, l(p) = l(q) \geq 1, n = l(p)$. If $g_w(n)^* t g_w(n) \neq 0$ then either $p = q = g_w(n)$ or $l(g_w(n)) = r < n$ and $r(g_w(n))$ is a sink.

The first case is impossible since $g_w(n)g_w(n)^* \notin B(\gamma)$. If $l(g_w(n)) = r, 1 \leq r < n$, then $g_w(n)^* p = q^* g_w(n)$ since the path $g_w(n)$ ends with a sink and therefore can not be a beginning of paths p, q. Finally, if w is a sink then it can not be the source of paths p, q. Hence, for a sufficiently large n we have $g_w(n)^* b g_w(n) = r(g_w(n)) + \sum \mu_{p,q} g_w(n)^* pq^* g_w(u)$. This expression is not equal to 0 by Lemma 6. In each summand $g_w(n)^* pq^* g_w(n)$ the special edges in p, q won't cancel. Denote $u = r(g_w(n))$. We have $0 \neq c = u + \sum \nu_{p,q} pq^* \in J_0$; $l(p) = l(q) \geq 1, sd(p) = sd(q) = n, r(p) = r(q)$, both p and q contain non special edges, $pq^* \in B(\gamma)$, for each summand. Now, as we did above, consider $g_u(m)^* c g_u(m) = r(g_u(n)) + \sum \nu_{p,q} g_u(m)^* pq^* g_u(m)$ and $g_u(m) g_u(m)^* c g_u(m) g_u(m)^* = g_u(m) g_u(m)^* + \sum \nu_{p,q} pq^*$, where both p, q in each summand are continuations of $g_u(m)$. If $r(g_u(m))$ is a sink then $g_u(m)^* c g_u(m) = r(g_u(m)) = e_{r(g_u(m))} \in J$. If for any $m \geq 1$, $r(g_u(m))$ is not a sink then the sequence $\sum \nu_{p,q} pq^*$, where p, q are continuations of $g_u(m)$, converges to 0 as $m \to \infty$. This implies $e_u = \lim_{m \to \infty} g_u(m) g_u(m)^* \in J$, and completes the proof of Proposition. □

Corollary 2 *The algebra $\widehat{L}(\Gamma)$ does not have non-zero graded nilpotent ideals.*

Proof Let I be a nonzero graded nilpotent ideal of $\widehat{L}(\Gamma)$. The graded closure $\bar{I}$ of the ideal I in $\widehat{L}(\Gamma)$ is again nilpotent. By Proposition 1 there exists a vertex $v \in V$ such that the idempotent e_v lies in $\bar{I}$, a contradiction. □

Lemma 23 *Let W be a minimal nonempty hereditary subset of V. Then the ideal $I(W)$ is generated (as an ideal) by all idempotents e_w, $w \in W$.*

Proof If W consists of one sink w then $e_w = w$. Suppose therefore that the subset W does not contain sinks. Let J be the ideal of $\widehat{L}(\Gamma)$ generated by all idempotents $e_w, w \in W, J \subseteq I(W)$.

For a vertex $w \in W$ we have $e_w = w - \sum \{g_w(k) e e^* g_w(k)^* \mid k \geq 0, e \in \mathcal{E}(r(g_w(k)))\}$. For arbitrary vertices $w, v \in W$, consider the operator $A_{w,v}$: $v\widehat{L}(r)v \to w\widehat{L}(r)w$,

$A_{w,v}(a) = \sum\{g_w(k)eae^*g_w(k)^* \mid k \geq 0, e \in \mathcal{E}(r(g_w(k))), r(e) = v\}$. If the vertex v does not appear as range of some path $g_w(k)e$, $e \in \mathcal{E}(r(g_w(k)))$, then $A_{w,v} = 0$.

Let $W = \{w_1, \ldots, w_r\}$. Consider the matrix $A = (A_{w_i,w_j})_{r\times r}$. Consider the $r-$tuples $\overline{w} = (w_1, \ldots, w_r)^T$ and $\overline{e}_w = (e_{w_1}, \ldots, e_{w_r})^T$. Then $\overline{e}_w = (I - A)\overline{w}$. We have $A^i\overline{w} \subseteq (V_{2i}, \ldots, V_{2i})^T$, hence $A^i\overline{w} \to 0$ as $i \to \infty$. Now , $\overline{w} = \sum_{i=0}^{\infty} A^i\overline{e_w} \in (J, \ldots, J)^T$, which proves the Lemma. □

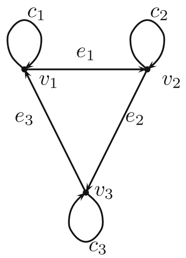
Corollary 3 *$\widehat{L}(\Gamma)$ is generated (as an ideal) by the set $\{e_w, w \in \bigcup_i W_i\}$.*

Let $V = S_1\dot{\cup}\ldots\dot{\cup}S_m$ be all connected components of the undirected graph $(V, \tilde{\gamma}(V))$. Let J_i be the closed ideal of $\widehat{L}(\Gamma)$ generated by the set e_v, $v \in S_i$.

Theorem 1 *Let Γ be a finite graph. Let γ be a frame specialization. Then*

1. $\widehat{L}(\Gamma) = J_1 \bigoplus \cdots \bigoplus J_m$;
2. *each J_i is a (topologically) simple algebra;*
3. $I(W_i) = \bigoplus J_i$, *the direct sum is taken over all J such that $S_J \cap W_i \neq \emptyset$.*

Proof The first assertion immediately follows from Lemma 21 and the corollary of Lemma 23. The second assertion follows from Proposition 1. The third assertion follows from Lemmas 21, 23, which finishes the proof of the Theorem. □

Example 15 Let $\Gamma =$ 

$\gamma(v_i) = e_i, 1 \leq i \leq 3$, the completion of $\widehat{L}(\Gamma)$ is the direct sum of 3 isomorphic simple summand: $\widehat{L}(\Gamma) = J_1 \oplus J_2 \oplus J_3$, where J_i is the (closed) ideal generated by the vertex v_i, $1 \leq i \leq 3$.

Remark that each component S_i intersects just one minimal hereditary subset W_i. Indeed, if $S_i \cap W_i \ni v$ and $S_i \cap W_j \ni w$, then by Lemma 20 the vertices v and w have a common descendant, which implies $i = j$. If $S_i \cap W_i = \emptyset$ for every i then by Lemma 21(2) we have $J_i.id(e_v, v \in \bigcup_i W_i) = (0)$, where $id(e_v, v \in \bigcup_i W_i)$ is the closed ideal generated by the elements e_v in $\widehat{L}(\Gamma)$. However, Lemma 23 implies that $id(e_v, v \in \bigcup_i W_i) = I(W_1) \bigoplus \cdots \bigoplus I(W_k) = \widehat{L}(\Gamma)$, a contradiction.

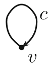
Now we will show that an arbitrary finite connected graph has a specialization in which the decomposition of the Theorem 1(3) looks particularly nice.

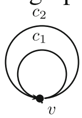
If $\gamma : V \to E$ is a specialization of a graph Γ and W is a hereditary subset of V then the restriction of γ to W is a specialization of the graph $(W, E(W, W))$. We will denote this restriction as γ_W.

Let $W_1, \ldots, W_k$ be the frame of the graph $\Gamma = (V, E)$.

Definition 9 We call a specialization $\gamma : V \to E$ regular if

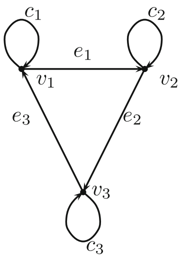
(1) γ is a frame specialization ,
(2) Each graph $(W_i, \tilde{\gamma}_{W_i})$ is connected, $1 \le i \le k$.

Example 16 The unique specialization of the graph 

is regular. Moreover, any choice of a special loop on the Leavitt graph 

is regular.

Example 17 The choice of the edge e as a special edge makes a regular specialization. However the specialization $\gamma(v) = c$ is not regular. Indeed, the only minimal hereditary subset of V is $\{w\}$. There are infinitely many special paths with all vertices lying in $V \backslash \{w\} = \{v\}$.

Example 18 Let $\Gamma =$ 

The graph $\Gamma = (V, E)$ does not have proper hereditary subsets. The specialization $\gamma(v_i) = e_i$, $1 \le i \le 3$, is regular . However, the specialization $\gamma(v_i) = c_i$, $1 \le i \le 3$, is not regular since the graph $(V, \tilde{\gamma}_V)$ is not connected.

Lemma 24 *An arbitrary finite graph* Γ *has a regular specialization.*

Proof By the proof of Lemma 13 arbitrary specializations of non-sink minimal hereditary subsets $\gamma_i : W_i \to E(W_i, W_i)$ can be extended to a specialization $\gamma : V \to E$ with the property (1). Hence, it remains to find regular specializations on graphs $(W_i, E(W_i, W_i))$, where W_i does not consist of one sink. We have already mentioned that each graph $(W_i, E(W_i, W_i))$ is strongly connected, that is every vertex of it is a descendant of every other vertex.

A graph (V, E) is called a tree if there exists a vertex $v_0 \in V$ such that an arbitrary vertex in V can be connected to v_o by a unique path. An arbitrary strongly connected graph (V, E) has a spanning subtree (V, E'), $E' \subseteq E$ (see [7]). Let (W_i, E_i) be a spanning subtree of the graph $(W_i, E(W_i, W_i))$, $E_i \subset E(W_i, W_i)$. Let $w_i \in W_i$ be a such a vertex that an arbitrary vertex in W_i can be connected to w_i by a unique path in (W_i, E_i).

If $w \in W_i$, $w \neq w_i$ then there exists a unique edge $e \in E_i$ such that $s(e) = w$. We let $\gamma_i(w) = e$. The edge $\gamma_i(w_i)$ is chosen arbitrarily in $s^{-1}(w_i)$. It is easy to see that the graph $(W_i, \tilde{\gamma}_i)$ is connected which finishes the proof of the Lemma. □

Now the Theorem 1 implies

Theorem 2 *If* γ *is a regular specialization then each ideal* $I(W_i)$ *is a topological graded simple algebra.*

Corollary 4 *If γ is a regular specialization then the Leavitt path algebra $L(\Gamma)$ is prime if and only if the completion $\widehat{L}(\Gamma)$ is topologically graded simple.*

Proof Indeed, if the Leavitt path algebra $L(\Gamma)$ is prime then the frame consists of one minimal hereditary subset W_1. By Theorem 2 the completion $\widehat{L}(\Gamma) = I(W_1)$ is topologically graded simple. On the other hand if the completion $\widehat{L}(\Gamma)$ is topologically graded simple then it is topologically graded prime. An arbitrary dense subalgebra in a topologically graded prime algebra is graded prime, hence prime. $\square$

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